

Quality factors of deformed dielectric cavities

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Abstract. An analysis is provided of the degradation that arises in the quality factor of a whispering gallery mode when a circular or spherical dielectric cavity is deformed. The large quality factors of such resonators are important to their use in applications such as sensors, wavelength filters or lasers. Yet a straightforward analysis of the effect of shape deformation on quality factors cannot be given because the underlying complex ray data demanded by a standard eikonal approximation frequently does not exist. In this paper we exploit an approach that has been successfully used elsewhere to describe the strong directional emission of such systems, based on a perturbative treatment of the relevant complex ray families. Applicable when the radial perturbation is formally of the order of a wavelength, the resulting approximation successfully describes changes to the quality factor using the ray geometry in a neighbourhood of a discrete set of escaping rays guiding the directions of maximum emitted intensity.

1. Introduction

Optical microcavities are widely exploited in applications such as lasers, sensors and wavelength filters [1, 2, 3, 4]. A key underlying feature in such technologies is that these devices support whispering gallery modes, which, corresponding to ray families confined by total internal reflection to the interior of the resonator, are very long lived. An important practical consideration is then to be able to characterise the quality factor of such systems. Geometry is a particularly important factor in this regard. Deformation from perfectly circular or spherical geometry, while leading to the often desirable feature of directional emission, typically degrades the quality factor itself [5, 6]. Here we treat the case of smooth shape deformations but note that qualitatively similar features may also be seen in microcavities perturbed by notches [7] or internal scatterers [8, 9, 10].

In this paper we focus particularly on the regime of weak deformation and offer analytical models for the quality factor in that limit. Whether deliberately engineered or arising as a consequence of manufacturing tolerances, even very weak deformations can be seen to have a dramatic effect on the external field of these systems and as a consequence (we will find) on the quality factor. Despite the fact that the internal modes and associated real ray dynamics are hardly qualitatively changed from the corresponding exact circular or spherical limit, a straightforward application of eikonal theory frequently fails for such systems because natural boundaries [11, 12, 13] may prevent one from calculating the complex ray data required by an approximation of the

external, evanescent field. Here we exploit an approach which has successfully been used elsewhere to characterise the directional emission of such systems [14, 15]. Rather than demand exact solutions of the eikonal equation governing ray dynamics, which natural boundaries may prevent us from continuing sufficiently far into the complex domain, we instead construct approximate ray families, calculated using canonical perturbation theory.

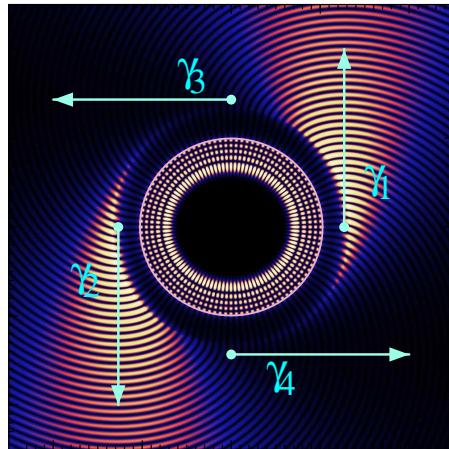


Figure 1. The external field is shown for a whispering gallery mode in an elliptical cavity with refractive index $n = 2$ and focal half-length $c = r_0/4$ where $ab = r_0^2$ is the area. This mode has azimuthal quantum number $m = 50$ and 5 radial nodes (with $k_r r_0 \approx 37.5$) and we note that the external field is exaggerated graphically to make it visible. Superimposed are the real escaping rays γ_1 and γ_2 guiding waves of maximum intensity and the real rays γ_3 and γ_4 guiding waves of minimum intensity. The geometry of the ray family around these particular escaping rays determines the Q factor.

Once the external field has been successfully approximated in this manner, the quality factor itself may be calculated using Herring's formula [16], which represents it as a flux integral measuring the overall rate of escape from the resonant mode. A straightforward evaluation of this flux integral, following a truncation to first order of a perturbative approximation of ray families, is shown in explicit numerical examples to describe significant degradations of the quality factor in problems where the deformation seems at first sight to be very weak. It should be emphasised that, although the underlying ray calculation is perturbative, it successfully described emitted waves that are strongly altered by the deformation and applies when perturbation of the wave solutions itself is not valid [17].

When approximated for somewhat larger deformations using saddle-point approximation, one finds that the dominant contribution to the Herring integral is associated with a discrete set of rays escaping to infinity along the directions of greatest emitted intensity (such as γ_1 and γ_2 in Fig. 1). Dynamically, the distinguishing feature of these rays is that they propagate with real direction cosines, whereas one finds generically in deformed resonators that escaping rays are slightly complex. In terms of phase-space geometry, the escaping ray family locally forms a complex Lagrangian manifold and the dominant real escaping rays form the intersection of this Lagrangian

manifold with its complex conjugate. For sufficiently large deformations, the leading order approximation of the quality factor is determined by a variant of Wilkinson's formula [18, 16], which expresses it in terms of canonically invariant measures if the transversality of this intersection.

Although a primitive application of the steepest descents approximation fails in the limit of very small perturbation, we also offer a uniform analysis allowing one to interpolate smoothly from the case of small deformation, where simpler perturbative results apply, to larger deformations where Wilkinson's formula can be used. This is shown in particular to give a good description of the quality factor of elliptical cavities, where the existence of exact invariants allows us to evaluate the relevant complex ray data for a wide range of deformations.

We conclude by summarising the content of the paper. We begin in Sec. 2 by setting out the basic problem to be solved and the notation used. We also describe in general terms how Herring's formula can be used to calculate the quality factor and outline the main approaches to its approximation in the cases of weak deformation, where completely perturbative results are appropriate, and of moderate deformation where the Herring flux integral may be tackled using approximation by the method of steepest descents. A more detailed description of these approaches, and their results, are offered in subsequent sections. Sec. 3 describes the perturbative approximation which may be applied to weak but generic deformations. Sec. 4 describes the asymptotic development of the Herring integral appropriate to larger deformations and applies this approach in particular to the special case of elliptic cavities, where the requisite complex ray data may be explicitly characterised, even for larger deformations. Finally, conclusions are offered in Sec. 5.

2. Notation, Herring's formula and its approximation

2.1. Notation

We consider a two-dimensional scalar model governed by the Helmholtz equation

$$-\nabla^2\psi = n^2(\mathbf{x})k^2\psi.$$

(We expect that the basic underlying approach offered here, and the general conclusions reached, will apply also to higher dimensional problems or vector formulations, but we restrict our detailed discussion to two-dimensional scalar waves for clarity of exposition.) Denote the region occupied by the cavity by Ω and its boundary by $\partial\Omega$, so that the refractive index changes discontinuously from a constant value $n > 1$ in Ω to $n = 1$ outside Ω . The resonances we treat are examples of Gamow-Siegert states [19, 20], and are solutions of the Helmholtz equation subject to outwards-radiating boundary conditions at infinity. Such resonances are found for discrete complex wavenumbers

$$k = k_r - \frac{i}{2}\kappa$$

with a negative imaginary part, which is small when the resonances are long-lived, as is the case for the whispering-gallery modes we consider. The quality factor

$$Q = \frac{k_r}{\kappa}$$

is then large. The Gamow-Siegert states are not normalisable and instead we adopt a normalisation convention

$$\|\psi\|_{\Omega}^2 \equiv \int_{\Omega} |\psi(\mathbf{x})|^2 d\mathbf{x} = 1, \quad (1)$$

in which integration is restricted to the cavity's interior.

2.2. Herring's integral

Herring's formula is obtained by integrating the identity

$$\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* = 2i\kappa k_r \psi^* \psi$$

over a region R containing Ω , and using Green's identities to get

$$\kappa = \frac{1}{2ik_r \|\psi\|_R^2} \int_{\Sigma} \left(\psi^* \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi^*}{\partial n} \right) ds, \quad (2)$$

where s represents arc-length along $\Sigma \equiv \partial R$ and where $\|\psi\|_R^2$ is defined as in (1), but with integration restricted to R instead of Ω . The result is, of course, independent of the choice of region R , although some restriction on this choice makes the approximation that follows easier. In particular, we choose the section Σ to be far enough outside Ω that the rays underlying an eikonal approximation of ψ are approximately real (see below), but close enough to Ω that the exponentially small contribution to $\|\psi\|_R^2$ from the exterior of Ω can be neglected, allowing us to approximate

$$\|\psi\|_R^2 \approx \|\psi\|_{\Omega}^2 = 1.$$

The Herring integral (2) then measures the flux of the quasinormalised state ψ escaping across the section Σ . We now outline how an asymptotic approximation of it is determined in terms of the geometry of the ray family underlying ψ .

2.3. Approximation of the wavefunction

In the case of whispering gallery modes in a circular cavity, the wave field immediately outside the resonator decays exponentially in the radial direction and is approximated by a family of rays whose radial momentum is purely imaginary. Far enough outside the resonator, however, a caustic is encountered beyond which the rays once again propagate with real radial momentum and the wavefunction is oscillatory, albeit with an exponentially small amplitude following the decay that occurs through the evanescent band immediately outside the resonator. We consider the case of deformed resonators sufficiently close to circular that a deformation of this ray family can be defined which provides an eikonal approximation

$$\psi(\mathbf{x}) \approx A(\mathbf{x}) e^{ikS(\mathbf{x})}$$

for the field in the exterior region. Although natural boundaries may prevent us from defining such a ray family exactly, we have shown elsewhere that approximate families can be constructed using canonical perturbation theory which successfully describe the deformed exterior wave field [14, 21]. We explain this construction in more detail later but for now we simply assume that such a ray family can be defined, with an overall geometry that is a deformation of the circular case.

We consider in particular the case where \mathbf{x} is far enough outside the resonator that the rays of the circular limit are real. In the deformed case, a symmetry breaking

is found to occur so that the corresponding rays become slightly complex. That is, even for $\mathbf{x} \in \mathbb{R}^2$, the rays propagate with complex direction cosines. Then the resulting direction dependence of the imaginary part $\text{Im}(S(\mathbf{x}))$ of the action function can strongly modulate the wave intensity $|\psi(\mathbf{x})|^2$ and one finds that the Herring integral may be dominated by a few directions of maximal flux density. We now argue that the condition of maximum intensity/flux corresponds in terms of ray geometry to the condition that the escaping rays are real, and provide a general description of the asymptotic evaluation of the flux integral around these real ray directions.

2.4. Approximation of the Herring integral

To describe the asymptotic evaluation of the Herring integral (2) more explicitly, let us write the amplitude of the wave field in the form

$$A(\mathbf{x}) = t\sqrt{\rho(\mathbf{x})},$$

where t is a transmission coefficient coupling the interior and exterior solutions, defined so that the remaining density term takes a standard WKB form. [We point out that a full WKB treatment of dielectric cavities requires the inclusion of a phase — related to the Goos-Hänchen shift [22, 23] — accounting for variation of the reflection phase along the boundary. To simplify the presentation as much as possible we assume here that any such phases have been incorporated into the transmission amplitude t .] Then, at leading order (in an expansion in powers of $1/k$), Herring's integral becomes

$$\kappa \approx \int_{\Sigma} \hat{\mathbf{n}} \cdot \left(\frac{\mathbf{p} + \mathbf{p}^*}{2} \right) \psi^* \psi \, ds \approx \int_{\Sigma} (\text{Re } p_n) |t|^2 |\rho| e^{-2k_r \text{Im} S} \, ds, \quad (3)$$

where $\hat{\mathbf{n}}$ is the unit normal to Σ , $\mathbf{p}(\mathbf{x}) = \nabla S$ and $p_n = \hat{\mathbf{n}} \cdot \mathbf{p}$.

Further simplification of this integral can be achieved by exploiting the associated dynamical invariants for modes with an integrable underlying ray dynamics. Any ray-dynamical system enjoys the invariant $H = (p_x^2 + p_y^2)/2$. An additional invariant, which we denote by M , is assumed to exist for the ray families underlying the modes we treat. This second invariant is angular momentum for the circular limit and a deformation of it for perturbed cavities. In the case of elliptical deformations, an explicit analytical form will be given for M in Sec. 4 whereas for the case of generic deformation, an approximate invariant M is defined implicitly by the canonical perturbation approach we employ in Sec. 3. Let χ denote an angle coordinate on Σ conjugate to M . That is, χ evolves at a constant rate under the flow generated by using M as a Hamiltonian and we scale it so that it has period 2π . Then, it can be shown that

$$|\rho| ds = \frac{\mathcal{J}}{2\pi} \frac{d\chi}{|p_n|},$$

where

$$\mathcal{J} = \left| \frac{\partial(\varphi_1, \varphi_2)}{\partial(t, \chi)} \right| = \left| \frac{\partial(H, M)}{\partial(J_1, J_2)} \right|$$

is a Jacobian relating the variables (t, χ) , respectively conjugate to (H, M) , to regular action-angle variables $(\varphi_1, \varphi_2, J_1, J_2)$ for the ray family inside the cavity. The important feature for now is that \mathcal{J} is constant on ray families. Herring's integral can now be written

$$\kappa \approx \frac{\mathcal{J}}{2\pi} \int_{\Sigma} \frac{\text{Re } p_n}{|p_n|} |t|^2 e^{-2k_r \text{Im} S} \, d\chi. \quad (4)$$

Significant further approximation of this integral is possible in two particular regimes.

First, in the case of small generic deformations on a scale $\varepsilon = O(1/k)$, the imaginary part of the action can be computed perturbatively while the amplitude variation can be neglected and the quantities p_n and t in this integral replaced by the constant values they take for the circular limit. Then

$$\kappa \approx \kappa_0 \mathcal{M}, \quad (5)$$

where

$$\mathcal{M} = \frac{1}{2\pi} \oint e^{-2k_r \text{Im}(S(\chi) - S_0)} d\chi, \quad (6)$$

where κ_0 and $S_0(\chi)$ respectively denote the value of κ and the action function for the circular limit. Further development of this approximation will be offered in Sec. 3 where approximations for $\text{Im}(S(\chi))$ are developed by applying canonical perturbation theory to the ray families. We also note that the integrand $e^{-2k_r \text{Im}(S(\chi) - S_0)}$ provides a leading-order measure of the intensity of the emitted field and has already been calculated in that context in [14]. The simple perturbative procedure pursued in this paper applies in the generic case where the unperturbed ray family is not too close to resonances of the ray dynamics. When ray-dynamical resonances are nearby, a somewhat more involved approximation of the exterior field is required but even in that case we note that the Q -factor degradation can be evaluated once the exterior field is known using

$$\mathcal{M} = \int |\psi_\varepsilon|^2 d\chi / \int |\psi_0|^2 d\chi$$

where ψ_0 and ψ_ε respectively denote the circular limit and the ε -deformed form of the exterior field. Explicit calculations are confined in this paper to the simpler case described by (6), however.

For larger deformations, it is natural to approximate the integral using the method of steepest descents. For generic deformations, however, the intervention of natural boundaries means that we often cannot find the required ray data. An exception is the case of elliptical cavities where a complete analytical description of the external ray family can be achieved. The resulting approximation for this case is described in detail in Sec. 4 but here we summarise the main qualitative features, which also apply to the large- εk asymptotics of (6). Stationary points of the exponent $-2\text{Im}(S) = i(S - S^*)$ in Herring's integral are obtained for real \mathbf{x} when

$$\mathbf{p} = \mathbf{p}^*, \quad (7)$$

that is, when the associated escaping rays are *real*. Generic escaping rays are complex for deformed resonators (albeit only slightly complex when the resonator is weakly eccentric). A discrete subset of real escaping rays can typically be found, however, propagating along the directions of greatest and least emitted intensity. A steepest-descents approximation of Herring's integral therefore characterises the decay rate in terms of the behaviour of the mode around the directions of greatest concentration of escaping flux, as one might expect intuitively. Further details of this approach are developed in Sec. 4.

3. Approximation of Herring's integral for generically perturbed cavities

In this section we describe how the weak-deformation approximation (5) of the Herring integral may be exploited to describe the degradation of the quality factor for boundary shapes of the form

$$r(\theta) = r_0 + \varepsilon r_1(\theta).$$

Recall that natural boundaries prevent a systematic calculation of escaping rays in generically deformed cavities but that, for small enough deformations, approximate ray families calculated using canonical perturbation theory can successfully describe the emitted field. We now summarise the main relevant results from that calculation and use them to evaluate the integral \mathcal{M} in (5).

3.1. Calculation of external field using perturbative approximation of rays

The first step in finding the complex action function $S(\mathbf{x})$ of escaping rays is to calculate the corresponding action function on the boundary $\partial\Omega$. This must be done for complex coordinates to allow a continuation to the farfield by tracing the associated complex escaping rays.

The action is found on $\partial\Omega$ by applying canonical perturbation theory to the internal dynamics, following the approach developed by Prange and Zaitsev in the context of billiard problems [24, 25]. Dynamics on the boundary is characterised by using a perturbative expansion of the chord function

$$L(\theta, \theta') = L_0(\theta, \theta') + \epsilon L_1(\theta, \theta') + \dots, \quad (8)$$

expressing, as a series in the perturbation parameter ε , the length of the chord connecting points on the boundary with polar angles θ' and θ . The leading term

$$L_0(\theta, \theta') = 2r_0 \sin \left| \frac{\theta - \theta'}{2} \right|$$

is the chord function for a circle of radius r_0 and the first-order term is

$$L_1(\theta, \theta') = \sin \left| \frac{\theta - \theta'}{2} \right| (r_1(\theta) + r_1(\theta')).$$

The chord function $L(\theta, \theta')$ serves as a type-one generating function for the boundary map, expressed in terms of the polar angle θ and its conjugate momentum variable $J = p_\theta$. Note that (θ, J) serve as action-angle variables for the unperturbed boundary dynamics. The action-angle variables $(\bar{\theta}, \bar{J})$ for the perturbed system can then be described by the type-two generating function

$$\hat{S}(\theta, \bar{J}) = \theta \bar{J} + \epsilon g(\theta, \bar{J}) + \dots, \quad (9)$$

where $g(\theta, \bar{J})$ is shown to satisfy the difference equation

$$g(\theta + \omega, \bar{J}) - g(\theta, \bar{J}) = R(\theta, \bar{J}), \quad (10)$$

where

$$R(\theta, \bar{J}) \equiv L_1(\theta + \omega, \theta) - \langle L_1 \rangle$$

denotes the oscillating part, with respect to θ , of $L_1(\theta + \omega, \theta)$, with $r_0 \cos(\omega/2) = \bar{J}$. Note that \bar{J} takes a fixed value determined by the quantisation condition $nk\bar{J} = m$, where m is the azimuthal quantum number of the unperturbed mode.

This difference equation for $g(\theta, \bar{J})$ is straightforwardly solved, as a Fourier series for example, for any analytic function $r_1(\theta)$. Note that the rays escaping to infinity typically start with complex values of θ . Here we limit, our perturbative expansion to a first-order truncation, in which case it suffices to substitute in $g(\theta, \bar{J})$ the initial conditions for the unperturbed escaping rays of the circle. For a counterclockwise-rotating mode, these can be shown to originate at $\theta_0 = \chi - \beta = \chi - \pi/2 + i\Theta$, where χ is the polar angle of a ray in the farfield and $\beta = \pi/2 - i\Theta$ is the angle of refraction with which it leaves the cavity's boundary. From Snell's law, $\cosh \Theta = \sin \beta = n \sin \alpha = nJ/r_0$, where α is the (constant) angle of reflection of the unperturbed internal rays. If the terms in the Fourier series for $r_1(\theta)$ decay slowly enough, the solution $g(\theta, \bar{J})$ of this Fourier series may have natural boundaries which lie below the initial conditions on $\text{Im}(\theta) = \Theta$ for escaping rays. In this case it is not known how to approximate external field arbitrarily far outside the resonator. We therefore restrict our attention to the case where $g(\theta, \bar{J})$ can be successfully extended in the complex plane as far as $\text{Im}(\theta) = \Theta$. This is true in particular when $r_1(\theta)$ is a trigonometric polynomial function of θ , which is the case for the models used in our numerical illustrations.

Next, the generating function $\hat{\mathcal{S}}(\theta, \bar{J})$ can also be shown to determine, at first order, the phase of the wave field on the boundary. That is, on the boundary, $\psi = B(\theta) e^{nik(\theta, \bar{J}) + \varepsilon g(\theta, \bar{J}) + \dots}$. The action along a ray emerging from θ with polar angle of propagation χ is then

$$S(\mathbf{x}, \bar{J}) = \ell(\mathbf{x}, \theta) + n\hat{\mathcal{S}}(\theta, \bar{J}) + O(\varepsilon^2)$$

where

$$\ell(\mathbf{x}, \theta) = \ell_0(\mathbf{x}, \theta) + \varepsilon \ell_1(\mathbf{x}, \theta) \dots$$

denotes the length of the ray connecting the point with polar angle θ on $\partial\Omega$ to the point \mathbf{x} on Σ and where, if required, θ can be expressed in terms of \mathbf{x} and \bar{J} .

Finally, as the cavity is deformed, the initial coordinate θ of the ray to a fixed exterior point changes. However, the functions $\ell(\mathbf{x}, \theta)$ and $\hat{\mathcal{S}}(\theta, \bar{J})$ can be shown respectively to satisfy the generating-function conditions $\partial\hat{\mathcal{S}}(\theta, \bar{J})/\partial\theta = r_0 \sin \alpha$ and $\partial\ell(\mathbf{x}, \theta)/\partial\theta = -r_0 \sin \beta$. By Snell's law, the effect on $S(\mathbf{x}, \bar{J})$ of changing θ therefore cancels at first order in ε and we may approximate

$$S(\mathbf{x}, \bar{J}) = S_0(\mathbf{x}, \bar{J}) + \varepsilon S_1(\mathbf{x}, \bar{J}) + O(\varepsilon^2),$$

where

$$S_1(\mathbf{x}, \bar{J}) = \ell_1(\mathbf{x}, \theta_0) + n g(\theta_0, \bar{J}) \quad (11)$$

and $S_0(\mathbf{x}, \bar{J})$ and θ_0 are, respectively, the action function and the launching angle for the unperturbed limit.

3.2. Calculating the Q -factor

Noting that rays in the farfield of the unperturbed problem satisfy $\theta_0 = \chi - \pi/2 + i\Theta$, the Q -factor degradation (6) can then be approximated

$$\mathcal{M} = \frac{1}{2\pi} \oint e^{-2\varepsilon k_r \text{Im} f(\chi)} d\chi.$$

where,

$$f(\chi) = S_1(\mathbf{x}(\chi), \bar{J})$$

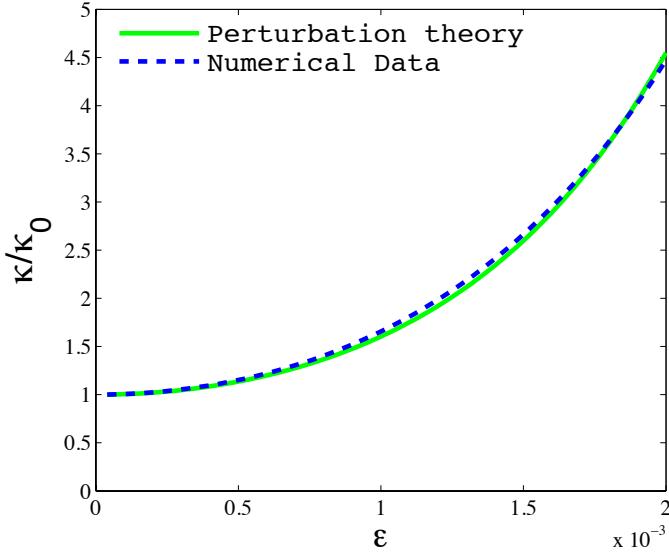


Figure 2. The perturbative approximation of κ in (12) is compared with a numerical evaluation for a deformation of the form $r = r_0(1 + \varepsilon \cos 3\theta)$ of the circle. The example shown here is a mode with azimuthal quantum number $m = 60$ and 6 radial nodes (with $k_r r_0 \approx 45.3$) in a cavity with refractive index $n = 2$.

is obtained by evaluating (11) at a position $\mathbf{x}(\chi)$ on Σ defined by the polar angle χ and using the unperturbed launch angle θ_0 .

To illustrate the calculation more explicitly we now consider deformations of the form

$$r_1(\theta) = r_0 \cos N\theta,$$

where N is an integer. The difference equation (10) has the solution

$$g(\theta, \bar{J}) = \frac{r_0 \sin(\omega/2) \sin N\theta}{\tan(N\omega/2)}$$

in this case, while

$$\ell_1(\mathbf{x}, \theta) = -r_1(\theta) \cos(\chi - \theta),$$

and it can be shown as a result that

$$\text{Im } f(\chi) = f_0 \cos N \left(\chi - \frac{\pi}{2} \right)$$

where

$$f_0 = \frac{n r_0 \sinh N\Theta \sin(\omega/2)}{\tan(N\omega/2)} - r_0 \cosh N\Theta \sinh \Theta$$

Then,

$$\mathcal{M} = I_0(2k\varepsilon f_0), \quad (12)$$

where $I_0(z)$ denotes a modified Bessel function of the first kind.

A typical implementation of this result is illustrated in Fig. 2. It should be emphasised that even though the perturbations involved here are quite small (typically of the order of one part in a thousand in a cavity that is about 30 wavelengths in diameter), the deformation is enough to significantly change the Q -factor — by a factor of 4 or more in the illustration. We also emphasise that, despite the apparent smallness of the perturbation, natural boundaries typically prevent us using a straightforward exact determination of the rays in this example.

4. Wilkinson's formula and the ellipse

For larger deformations, where the relevant eikonal phase is a rapidly-varying function of its arguments, Herring's integral (2) invites approximation by the method of steepest descents. This approach is attractive because it provides a direct interpretation of the Q -factor degradation in terms of simple geometrical properties of particular escaping rays — we will see in fact that it takes the form of a formula developed by Wilkinson in the context of tunnel splittings of quantum-mechanical energy levels [16].

For generic deformations such as treated in the previous section, Wilkinson's formula provides us with a simple ray-geometrical interpretation of the large- εk asymptotics of approximations such as (12). It should be acknowledged, however, that the method will in that case be restricted to moderately large values of εk where the perturbative approach remains valid. The asymptotic approach then has the benefit primarily of providing a useful ray-geometric interpretation of existing calculations rather than pushing them into fundamentally different regimes.

The case of elliptical deformation, however, is a particularly important special case where significantly greater deformations *are* treatable and where the asymptotic approach has the added value of extending the results of the previous section to different regimes.

We begin by describing the appropriate generalisation of Wilkinson's formula to the calculation of Q -factor degradation in a more general context where the escaping rays (or some approximation of them) are assumed to be known but without making further assumptions. We then apply this result to the specific case of elliptical cavities where the known invariants of the underlying ray families allow us to provide an explicit evaluation of the general result. Finally, we describe at the end of this section how a uniform evaluation of Herring's integral allows us to interpolate smoothly between the generic perturbative results of the previous section and the large-deformation limit described by Wilkinson's formula.

4.1. Wilkinson's formula in general

Wilkinson's formula is obtained by carrying out the steepest-descents approximation of Herring's integral outlined in section 3. Recall that we assume the existence (exactly, as in the ellipse, or more generally following a perturbative approximation) of an invariant M of the ray family in addition to the usual ray invariant $H = (p_x^2 + p_y^2)/2$ and that the steepest-descents condition (7) is satisfied by a discrete set of real escaping rays guiding the maximum-intensity waves to infinity.

Then expansion of (3) (or, equivalently, (4)) about each of these stationary points

leads to the approximation

$$\kappa = \sum_{\gamma} (\text{Re } p_n) |t|^2 |\rho| \left| \frac{k_r}{2\pi i} \frac{\partial^2 (S - S^*)}{\partial s \partial s} \right|^{-1/2} e^{-2k_r \text{Im } S}, \quad (13)$$

where γ labels the real rays escaping to infinity along directions of maximal intensity. Further manipulation of the amplitude following techniques given in [26] enables it to be expressed in the following canonically invariant form:

$$\kappa = \sum_{\gamma} \frac{|t|^2 \mathcal{J}}{(2\pi)^{3/2} k_r^{1/2}} \frac{e^{-2k_r \text{Im } S}}{\sqrt{|\{M, M^*\}|}}. \quad (14)$$

The term in the square root here denotes a Poisson bracket between the invariant M and its complex conjugate M^* . This Poisson bracket necessarily vanishes on real ray families, for which $M = M^*$, so the denominator in Wilkinson's formula effectively provides a (canonically invariant) measure of the rate at which rays become complex as one moves away from the particular real rays labelled by γ . Alternatively it measures the transversality of the ray family to its complex-conjugate partner around their real intersection along γ . We also note that, using the fact that M and M^* each Poisson commute with $H = (p_x^2 + p_y^2)/2$, the Jacobi identity yields $\{\{M, M^*\}, H\} = -\{\{M^*, H\}, M\} - \{\{H, M\}, M^*\} = 0$. Because the flow of H is along rays, this means that $\{M, M^*\}$ is invariant along each real ray γ and it therefore doesn't matter where on γ we calculate it.

4.2. Wilkinson's formula for the ellipse

As described in [15], the external field can be described in detail in the special case of elliptical deformations by exploiting the existence of explicit analytical expressions for the invariant M for that case. We now use these results to offer a detailed evaluation of Wilkinson's formula for the dielectric ellipse.

We begin by summarising the main geometric characteristics of the rays of the dielectric ellipse, with a more detailed description being available in [15]. We denote by a and b the major and minor semiaxes and by c satisfying $a^2 + b^2 = c^2$ the half-distance between foci. Ray families in the ellipse's interior have, as an invariant,

$$A(\mathbf{x}, \mathbf{p}) = (xp_y - yp_x)^2 - c^2 p_y^2,$$

which expresses in cartesian coordinates the product of angular momenta about the two foci. The invariant in the exterior takes a different functional form but can be expressed in terms of the function $A(\mathbf{x}, \mathbf{p})$ by matching interior and exterior families using Snell's law. Our convention is that the exterior invariant M thus obtained is scaled so that

$$n^2 M = A - (n^2 - 1)c^2 \sin^2 u. \quad (15)$$

Here, u denotes the boundary coordinate from which an external ray is launched, where in general u parameterises the ellipse boundary according to $(x, y) = (a \cos u, b \sin u)$. Because rays guiding the external wave to infinity have complex starting points on the boundary, u and therefore M , are complex functions of the external phase space coordinates (\mathbf{x}, \mathbf{p}) . In fact, more detailed calculations [15] show that

$$\sin u = \frac{-bp_x L + iap_y Q}{b^2 + c^2 p_y^2},$$

where $L = xp_y - yp_x$ denotes angular momentum about the centre of the ellipse, and $Q \equiv (A - b^2(p_x^2 + p_y^2))^{1/2}$. We can then separate the real and imaginary parts of the invariant M according to

$$n^2M = A - B^2 + C^2 - 2iBC, \quad (16)$$

where

$$B = -c\sqrt{n^2 - 1} \frac{bp_x L}{b^2 + c^2 p_y^2} \quad \text{and} \quad C = c\sqrt{n^2 - 1} \frac{ap_y Q}{b^2 + c^2 p_y^2}.$$

In order to implement (14) we must now find the real escaping rays γ , along with their complex starting points on the boundary, and then evaluate their actions and amplitudes in (14).

Assuming foci on the x -axis, we find that real rays guiding waves of maximum intensity emerge vertically and have starting points with u -coordinates

$$u_1 = iU_1 \quad \text{and} \quad u_2 = \pi + iU_1, \quad \text{where} \quad \sinh U_1 = \sqrt{\frac{n^2 M - b^2}{b^2 + n^2 c^2}}.$$

The full coordinates of these starting points are $(x_0, y_0) = (\pm a \cosh U_1, \pm b \sinh U_1)$ and the corresponding rays emerge (γ_1 and γ_2 in Fig. 1) with momentum $\mathbf{p} = (0, \pm 1)$, so that a general point on them has coordinates $(x, y) = (\pm a \cosh U_1, y_0 + t)$, $t \in \mathbb{C}$. The imaginary part of the action is therefore the imaginary part of the y displacement needed to get to real coordinate space, which is

$$\text{Im}S = p_y \text{Im}(y_0) = b \sinh U_1.$$

It remains to evaluate the Poisson bracket in the denominator of (14), which, from (16), takes the form.

$$n^4\{M, M^*\} = 4i\{A - B^2 + C^2, BC\}.$$

Evaluated on the dominant, vertically escaping rays, this can be shown after further manipulation to take the value

$$i\{M, M^*\}_{u_1, u_2} = \frac{8(n^2 - 1)c^2 b}{n^2 a^2} (M + c^2) \sqrt{\frac{n^2 M - b^2}{b^2 + n^2 c^2}}. \quad (17)$$

The final ingredient needed to evaluate (14) is the transmission coefficient, which, following the discussion in [15], can be shown to take the form

$$t = \frac{2\sqrt{n \sin \alpha \cos \beta}}{\cos \alpha + \cos \beta} e^{i\sigma},$$

where α and β are respectively the angles of incidence and reflection at the boundary point from which the escaping ray is launched. Our convention in this paper is that this transmission coefficient also incorporates a phase correction σ , described in detail in [15], that accounts for the variable reflection phases on the boundary and is related to the Goos-Hänchen shift [22, 23].

A numerical illustration of this result is given in Fig. 3. Also shown in the same figure is the perturbative prediction developed in the previous section. Approximation (14) is seen to give a good description of the quality factor degradation, and it works well for deformations strong enough that the perturbative result (12) has failed. Approximation (14) fails, however, for very small perturbations where, in the integrable circular limit in which all escaping rays become real, the Poisson bracket $\{M, M^*\}$ vanishes and the amplitude of (14) diverges. This is because in the regime of small $k\varepsilon$ the steepest descents approximation used to approximate κ is no longer justified. Here the perturbative approach of the previous section must be used instead. In the next subsection we describe how the two limits may be married in a single, uniform approximation that is valid in both limits.

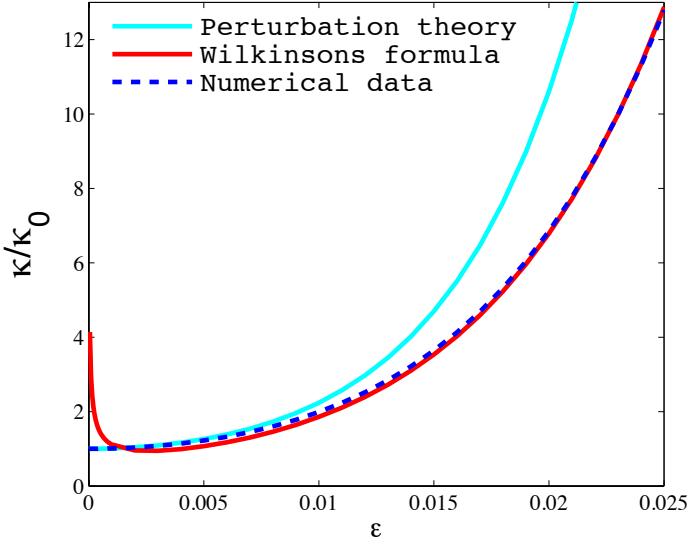


Figure 3. A comparison is given between calculations of κ using Wilkinson's formula (14), the perturbative approach of the previous section, and a numerical evaluation. The mode is the same as used in Fig. 1 and the perturbation parameter ε is defined so that $b = r_0(1 - \varepsilon)$ (so that $c^2 = a^2 - b^2 \approx 4\varepsilon r_0^2$).

4.3. Uniform approximation

A uniform approximation of the Q -factor degradation that interpolates smoothly between the perturbative approximation of Sec. 3 and Wilkinson's formula in (14) is now derived. The method here follows a uniform treatment in [27, 28] of the contribution of bifurcating periodic orbits to the trace formula.

We return to the version of Herring's integral given in (4) and seek a coordinate transformation $s \rightarrow \chi$ which is so that the imaginary action takes the form

$$\text{Im}S = \bar{K} + \Delta K \cos 2\chi,$$

where \bar{K} and ΔK are constants. The form chosen here exploits the symmetry of the problem with respect to $u \rightarrow u + \pi$ and the variable χ thus defined coincides at leading order with the polar angle used in the perturbative Herring integral (5), but deviates from it at higher deformation.

The constants \bar{K} and ΔK are determined by evaluating the minimum and maximum values $K_{\min} = \bar{K} - \Delta K$ and $K_{\max} = \bar{K} + \Delta K$ of $\text{Im} S$. We have already established that the minimum value $K_{\min} = b \sinh U_1$ is achieved by the real rays escaping along the directions of greatest intensity. The maximum value K_{\max} achieved by real rays (γ_3 and γ_4 in Fig. 1) escaping horizontally along directions of least intensity. It can be shown that these latter rays are launched from points

$$u = \pm \frac{\pi}{2} + iU_2, \quad \text{where} \quad \cosh U_2 = \sqrt{\frac{n^2 M}{a^2 - n^2 c^2}}$$

and have an imaginary action $\text{Im} S = K_{\max} = a \sinh U_2$.

Following this change of coordinates, the Herring integral is left in the form

$$\kappa = \frac{1}{2\pi} \int_0^{2\pi} \tilde{A}(\chi) e^{-k_r(\bar{K} + \Delta K \cos 2\chi)} d\chi. \quad (18)$$

We next write

$$\tilde{A}(\chi) = A_0 + A_1 \cos 2\chi + H(\chi) \frac{\partial}{\partial \chi} \Delta K \cos 2\chi, \quad (19)$$

where A_0 and A_1 are constants, chosen so that

$$H(\chi) = \frac{A_0 + A_1 - \tilde{A}(\chi)}{2\Delta K \sin 2\chi}$$

is smooth, which (exploiting the symmetry of the problem with respect to $u \rightarrow u + \pi$) is true provided

$$\tilde{A}(0) = A_0 + A_1 \quad (20)$$

and

$$\tilde{A}\left(\frac{\pi}{2}\right) = A_0 - A_1. \quad (21)$$

Substitution of (19) in (18), allows us, following integration by parts, to neglect at leading order in $1/k_r$ the contribution of the last term in (19). The result is

$$\kappa \approx (A_0 I_0(k_r \Delta K) + A_1 I_1(k_r \Delta K)) e^{-k_r \bar{K}}, \quad (22)$$

where $I_0(z)$ and $I_1(z)$ denote modified Bessel functions of the first kind.

In practice, the amplitudes A_0 and A_1 are conveniently evaluated by comparing the large-deformation asymptotics of (22) with the amplitude terms of Wilkinson's formula. A straightforward comparison yields

$$(A_0 + A_1) \sqrt{\frac{\pi}{k_r \Delta K}} = \frac{1}{(2\pi)^{3/2} k_r^{1/2}} \frac{|t(u_1)|^2 \mathcal{J}}{\sqrt{|\{M, M^*\}_{u_1}|}}.$$

Although the physical problem insists that k_r is positive, we might also formally compare asymptotic results for negative k_r and this yields

$$(A_0 - A_1) \sqrt{\frac{\pi}{k_r \Delta K}} = \frac{1}{(2\pi)^{3/2} k_r^{1/2}} \frac{|t(u_3)|^2 \mathcal{J}}{\sqrt{|\{M, M^*\}_{u_3}|}}.$$

The right-hand side here is determined by the geometry of the ray family around real rays escaping along the directions of *least* emitted intensity. Here we need the dark analogue of (17):

$$i\{M, M^*\}_{u_3, u_4} = -\frac{8(n^2 - 1)c^2 a}{n^2 b^2} M \sqrt{\frac{n^2 M - a^2 + n^2 c^2}{a^2 - n^2 c^2}}. \quad (23)$$

A_0 and A_1 are then known and the evaluation of (22) is complete.

This result is compared with numerical data in Fig. 4 and is seen to give a very good description of the Q -factor degradation from the perturbative regime of (12) to the larger-deformation regime of (14). In practice, the deformations treatable by this result are limited by our ability (or otherwise) to find phases σ — which have been absorbed in the transmission factor $|t|^2$ in this paper — which account for the dielectric boundary conditions in this problem. Such phases have been seen in [15] to suffer from natural boundaries which approach the real axis as eccentricity increases and invalidate the WKB ansatz above a critical deformation in the region of the complex boundary from which escaping rays are launched. It should be emphasised that the eccentricities at which this happens are formally of $O(1)$ and much greater than the eccentricities at which the perturbative result fails.

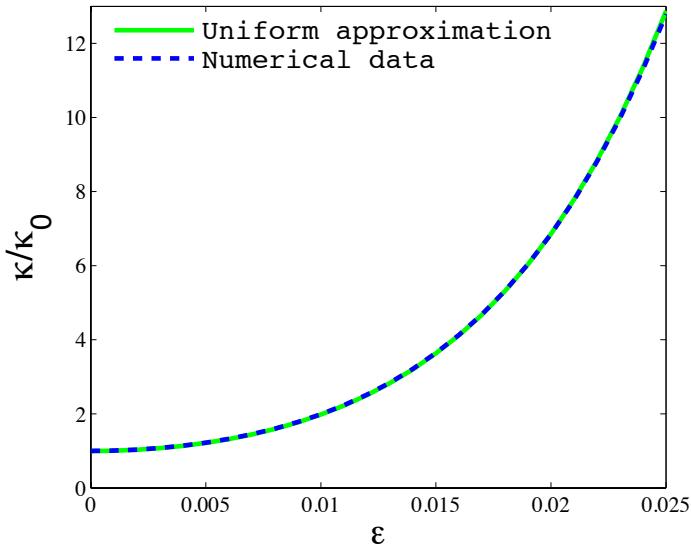


Figure 4. A comparison is given between calculations of κ using the uniform result (22) and a numerical evaluation for the same mode and parameter ranges used in Fig. 3.

5. Conclusions

We have found that, in a regime of short wavelength, very small deformations of optical cavities suffice to alter significantly the quality factor of resonant modes and have given quantitative estimates of this effect. We have shown that the quality factor may be halved or worse by deformations as small as one part in a thousand, for example, in explicit calculations where the cavity is some tens of wavelengths in diameter.

Although complex WKB methods provide a natural starting point for such calculations, their implementation is problematic because the required ray data may be difficult to find, or may not even exist. Instead, we have provided a general analysis based on approximation of the underlying ray families using perturbation techniques. Although restricted to relatively weak deformations, because the perturbations are applied to the rays rather than directly to the wave solution itself, this approach can nevertheless successfully describe deformations where the wave and quality factor itself are altered nonperturbatively.

For moderately large deformations, the results of this analysis are naturally interpreted in terms of the geometry of families of escaping rays, using Wilkinson's formula (14). Deformation effects the important qualitative change of making the associated family of escaping rays slightly complex (whereas far enough away from the cavity the corresponding orbit families are real in the undeformed case). Nevertheless a discrete subset of the escaping rays remain real and these in particular determine the directions along which the waves of greatest intensity propagate. Wilkinson's formula expresses the quality factor degradation in terms of the geometry of the ray family around the real subset guiding waves of highest intensity. It fails in the limit of small deformation where a denominator, which measures the rate at which rays

become complex away from the real subset, vanishes. A uniform result has been also derived, however, which interpolates between the primitive Wilkinson formula appropriate to larger deformations and the perturbative analysis derived earlier for weak deformations. This was shown to give an excellent description of the quality factor in the special case of elliptical deformations, where the ray families can be calculated analytically even for significantly eccentric cavities.

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